ELSEVIER

Journal of Computational and Applied Mathematics 111 (1999) 217–226

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.nl/locate/cam

A boundary integral method applied to a convection–diffusion problem

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Received 27 April 1998; received in revised form 15 January 1999

Abstract

A singular perturbed convection–diffusion problem on polygons is considered. Several boundary integral equations are used for the numerical approximation of the problem. An estimate for the single-layer integral operator in dependence of the perturbation parameter is obtained. The convergence of Galerkin and of Galerkin–Petrov schemes are discussed. Numerical results are presented. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 65R20; 65N38

Keywords: Convection–diffusion; Singular perturbation; Boundary integral equation; Galerkin and Galerkin–Petrov scheme

1. Introduction

In this paper we study the application of the boundary element method to the convection–diffusion problem

$$-\varepsilon\Delta u + 2\langle b, \nabla u\rangle = 0, \quad u|_\Gamma = f. \tag{1.1}$$

We shall assume that $\Omega \subset \mathbb{R}^2$ is a polygonal domain with boundary Γ, $b=(b_1,b_2)\in\mathbb{R}^2$, $f\in H^{1/2}(\Gamma)$, and $\varepsilon > 0$ are given. We look for a weak solution $u\in H^1(\Omega)$ of the boundary value problem (1.1). Problem (1.1) and more general elliptic equations have been studied with respect to the phenomena of singular perturbation and asymptotic expansions already in 1973 and before, see [7, Chapter 5] and the literature cited therein. The numerical solution of (1.1) and its generalisation to higher dimensions, parabolic equations and nonlinear equations was the subject of many papers, see [10]. All authors of these papers used finite element, finite differences or spectral element methods for the

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PII: S0377-0427(99)00144-2

numerical scheme. The use of boundary integral equations which have been successfully applied to a wide range of linear partial differential equations seems not to be studied extensively. We shall outline an approach here. First, we derive some results which lead in a well-known way, see for example [3], to the well-posedness of the boundary integral equations. These we shall formulate in Section 2 and obtain estimates for the Galerkin error in dependence of the parameter ε. We use standard Galerkin methods defined by piece-wise constant elements on the boundary. Our main purpose is to demonstrate some of the difficulties which appear in the numerical realisation of the singular perturbed problems in the case of boundary integral equations.

There should be several improvements possible of the scheme which we present here. One may try to define stabilised Galerkin methods, one may use the *hp*-method, wavelets, panel clustering, the multigrid technique or adaptive mesh refinements in combination with the boundary integral method. Nevertheless, the behaviour of these methods, which are well established for moderate values of ε seem not yet to be studied for the singular perturbed case of $\varepsilon \to 0^+$ in (1.1) and may be the subject for future work.

In Section 2 we define the boundary integral equations which we use for numerical approximation of Eq. (1.1). In Section 3 we obtain some analytic estimates which imply the coerciveness of the boundary integral operators and which are useful to obtain explicit constants in error estimates for the numerical schemes. In Section 4 we describe the implementation of the numerical schemes which includes the numerical cubature being used for computation of the elements of the stiffness matrices of the linear systems. In Section 5 we present the results of several numerical test computations.

2. The boundary integral equations

We use the substitution

$$u(x) = \mathrm{e}^{\langle b,x\rangle/\varepsilon} v(x), \quad x \in \Omega$$

to obtain the following boundary value problem for the Helmholtz equation which is equivalent to Eq. (1.1):

$$-\Delta v + \frac{|b|^2}{\varepsilon^2} v(x) = 0, \quad v|_\Gamma = \mathrm{e}^{-\langle b,x\rangle/\varepsilon} f(x). \tag{2.1}$$

The function $\mathrm{e}^{-\langle b,x\rangle/\varepsilon}$ is smooth with respect to $x \in \mathbb{R}^2$ and therefore we obtain that $\mathrm{e}^{-\langle b,x\rangle/\varepsilon} f(x) \in H^{1/2}(\Gamma)$. The Dirichlet problem (2.1) can be reduced to a boundary integral equation by the use of the fundamental solution of the Helmholtz equation $(1/2\pi)K_0(|b|/\varepsilon|x|)$. The representation formula for the solution $v \in H^1(\Omega)$ is given by, see [3]

$$v(x) = \frac{1}{2\pi} \int_\Gamma K_0\left(\frac{|b|}{\varepsilon}|x-y|\right)(\partial_{n_y} v)(y)\,\mathrm{d}s_y - \frac{1}{2\pi}\int_\Gamma \partial_{n_y}\left(K_0\left(\frac{|b|}{\varepsilon}|x-y|\right)\right) v(y)\,\mathrm{d}s_y, \quad x \in \Omega. \tag{2.2}$$

We shall use the single-layer potential ansatz

$$u(x) = \frac{1}{2\pi}\int_\Gamma \mathrm{e}^{\langle b,x-y\rangle/\varepsilon} K_0\left(\frac{|b|}{\varepsilon}|x-y|\right) g_1(y)\,\mathrm{d}s_y, \quad x \in \Omega \tag{2.3}$$

for the solution of Eq. (1.1) and we shall also use the single-layer potential ansatz for the solution of (2.1), namely, we represent v by

$$v(x) = \frac{1}{2\pi} \int_\Gamma K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) g_2(y) \, \mathrm{d}s_y, \quad x \in \Omega. \tag{2.4}$$

From representation (2.3) and the boundary condition in Eq. (1.1) we obtain that each solution g_1 of the integral equation

$$\frac{1}{2\pi} \int_\Gamma \mathrm{e}^{\langle b, x-y \rangle / \varepsilon} K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) g_1(y) \, \mathrm{d}s_y = f(x), \quad x \in \Gamma \tag{2.5}$$

leads to a solution u of (1.1). From the representation (2.4) and the boundary condition in Eq. (2.1) we obtain that g_2 has to solve the integral equation

$$\frac{1}{2\pi} \int_\Gamma K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) g_2(y) \, \mathrm{d}s_y = \mathrm{e}^{-\langle b/\varepsilon, x \rangle} f(x), \quad x \in \Gamma. \tag{2.6}$$

From representation (2.2) and the boundary condition in Eq. (2.1) we obtain that

$$\begin{aligned} &\frac{1}{2\pi} \int_\Gamma K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) (\partial_{n_y} v)(y) \, \mathrm{d}s_y \\ &= \frac{f(x) \mathrm{e}^{-\langle b/\varepsilon, x \rangle}}{2} + \frac{1}{2\pi} \int_\Gamma \left\{ \partial_{n_y} K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) \right\} f(y) \mathrm{e}^{-\langle b/\varepsilon, y \rangle} \, \mathrm{d}s_y, x \in \Gamma. \end{aligned} \tag{2.7}$$

3. Estimates for the single-layer potential operator and for the numerical schemes

Let $g \in H^{-1/2}(\Gamma)$. Then a distribution $g \otimes \delta_\Gamma \in H^{-1}(\mathbb{R}^2)$ is defined by, see [3], $(g \otimes \delta_\Gamma, u) = \int_\Gamma g \gamma_0 u$ for all $u \in H^1(\mathbb{R}^2)$ with trace $\gamma_0 u \in H^{1/2}(\Gamma)$ and the integral has to be understood in the sense of the natural pairing between $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, see [5] for the definition of the Sobolev spaces. The single-layer potential operator for the Helmholtz equation can be represented by the use of that distribution, namely,

$$\begin{aligned} A_\varepsilon g(x) &= \frac{1}{2\pi} \int_{\mathbb{R}^2} K_0 \left(\frac{|b|}{\varepsilon} |x - y| \right) (g \otimes \delta_\Gamma)(y) \, \mathrm{d}y \\ &= F^{-1} \frac{1}{|\xi|^2 + |b|^2/\varepsilon^2} F(g \otimes \delta_\Gamma) : H^{-1/2}(\Gamma) \to H^1(\mathbb{R}^2). \end{aligned}$$

In this equation we denote by Ff the two-dimensional Fourier-transform of f in the distributional sense, see [4]. A_ε is a convolution operator and $\{|\xi|^2 + |b|^2/\varepsilon^2\}^{-1}$ is the Fourier-transform of its kernel function, see [6, p. 134]. In the following, we shall denote by c a positive generic constant, which may depend on Γ, but not on the parameters ε, b_1, b_2. By an application of the trace lemma and Plancherel's theorem for the Fourier-transformation we obtain that

$$\begin{aligned} ||\gamma_0 A_\varepsilon g||^2_{H^{1/2}(\Gamma)} &\leqslant c ||A_\varepsilon g||^2_{H^1(\mathbb{R}^2)} \\ &= c \int_{\mathbb{R}^2} (1 + |\xi|^2) |F A_\varepsilon g(\xi)|^2 \, \mathrm{d}\xi = c \int_{\mathbb{R}^2} \frac{(1 + |\xi|^2)}{(|\xi|^2 + |b|^2/\varepsilon^2)^2} |F(g \otimes \delta_\Gamma)|^2 \, \mathrm{d}\xi \end{aligned}$$

$$\leqslant c \max\left(\frac{\varepsilon^4}{|b|^4}, 1\right) \int_{\mathbb{R}^2} \frac{1}{1+|\xi|^2} |F(g \otimes \delta_\Gamma)|^2 \, \mathrm{d}\xi$$

$$\leqslant c \max\left(\frac{\varepsilon^4}{|b|^4}, 1\right) ||A_1 g||^2_{H^1(\mathbb{R}^2)} \leqslant c \max\left(\frac{\varepsilon^4}{|b|^4}, 1\right) ||g||^2_{H^{-1/2}(\Gamma)}. \tag{3.1}$$

We get the following dependence of the constant in the coercivity estimate upon ε:

$$\frac{1}{2\pi} \int_\Gamma \int_\Gamma K_0\left(\frac{|b|}{\varepsilon}|x-y|\right) g(x) g(y) \, \mathrm{d}s_x \, \mathrm{d}s_y$$

$$= (g \otimes \delta_\Gamma, A_\varepsilon g) = \int_{\mathbb{R}^2} \frac{|F(g \otimes \delta_\Gamma)(\xi)|^2}{|\xi|^2 + |b|^2/\varepsilon^2} \, \mathrm{d}\xi \geqslant \min\left(\frac{\varepsilon^2}{|b|^2}, 1\right) \int_{\mathbb{R}^2} \frac{|F(g \otimes \delta_\Gamma)(\xi)|^2}{1+|\xi|^2} \, \mathrm{d}\xi$$

$$\geqslant \frac{1}{2\pi} \min\left(\frac{\varepsilon^2}{|b|^2}, 1\right) \int_\Gamma \int_\Gamma K_0(|x-y|) g(x) g(y) \, \mathrm{d}s_x \, \mathrm{d}s_y = \min\left(\frac{\varepsilon^2}{|b|^2}, 1\right) ||g \otimes \delta_\Gamma||^2_{H^{-1}(\mathbb{R}^2)}. \tag{3.2}$$

In the last step we take into account the coercivity of the single-layer integral operator for $\varepsilon = 1$. To outline this proof we introduce the operator T which maps each function $f \in H^{1/2}(\Gamma)$ to the unique weak solution $U = Tf \in H^1(\mathbb{R}^2 \setminus \Gamma)$ of the Dirichlet boundary value problem

$$-\Delta U + U = 0 \quad \text{in } \mathbb{R}^2 \setminus \Gamma, \quad U|_\Gamma = f.$$

The coercivity of the single-layer integral operator follows from

$$||g||_{H^{-1/2}(\Gamma)} \leqslant c \inf_{0 \neq f \in H^{1/2}(\Gamma)} \frac{|\int_\Gamma g(\gamma_0 Tf)|}{||f||_{H^{1/2}(\Gamma)}} = c \inf_{0 \neq f \in H^{1/2}(\Gamma)} \frac{|(g \otimes \delta_\Gamma, (Tf))|}{||f||_{H^{1/2}(\Gamma)}}$$

$$\leqslant c||g \otimes \delta_\Gamma||_{H^{-1}(\mathbb{R}^2)} \inf_{0 \neq f \in H^{1/2}(\Gamma)} \frac{||Tf||_{H^1(\mathbb{R}^2)}}{||f||_{H^{1/2}(\Gamma)}} \leqslant c||g \otimes \delta_\Gamma||_{H^{-1}(\mathbb{R}^2)}. \tag{3.3}$$

As a consequence, we obtain invertibility of the single layer integral operator for all $\varepsilon > 0$. Assuming $0 < \varepsilon < 1$, and $|b| = 1$ we obtain the following estimate for standard Galerkin methods when applied to Eqs. (2.5) or (2.7) and when defined by the use of a sequence of subspaces S^h approximating $H^{-1/2}(\Gamma)$, for example, piecewise polynomials on the boundary. By use of (3.2) we obtain the inequality

$$||g - g_h||^2_{H^{-1/2}(\Gamma)} \leqslant \frac{c}{\varepsilon^2}((g - g_h) \otimes \delta_\Gamma, A_\varepsilon(g - g_h)).$$

Using the definition of the Galerkin projector we get the identity

$$((g_h - \phi_h) \otimes \delta_\Gamma, A_\varepsilon(g - g_h)) = 0$$

for all $\phi_h \in S^h$, which together with the foregoing estimate implies the result

$$||g - g_h||^2_{H^{-1/2}(\Gamma)} \leqslant \frac{c}{\varepsilon^2}((g - \phi_h) \otimes \delta_\Gamma, A_\varepsilon(g - g_h)).$$

Taking into account the duality of $H^{-1}(\mathbb{R}^2)$ and $H^1(\mathbb{R}^2)$ the boundedness of the operator $\gamma_0 A_\varepsilon : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma)$ and taking into account the continuity of the mapping of $g - \phi_h \in H^{-1/2}(\Gamma)$ to

$(g-\phi_h)\otimes\delta_\Gamma\in H^{-1}(\mathbb{R}^2)$ we obtain the inequality

$$||g-g_h||^2_{H^{-1/2}(\Gamma)}\leqslant\frac{c}{\varepsilon^2}||\gamma_0 A_\varepsilon(g-g_h)||_{H^{1/2}(\Gamma)}||g-\phi_h||_{H^{-1/2}(\Gamma)}\leqslant\frac{c}{\varepsilon^2}||g-g_h||_{H^{-1/2}(\Gamma)}||g-\phi_h||_{H^{-1/2}(\Gamma)}$$

for all $\phi_h\in S^h$. Therefore, we proved the estimate

$$||g-g_h||_{H^{-1/2}(\Gamma)}\leqslant\frac{c}{\varepsilon^2}\inf_{\phi_h\in S^h}||g-\phi_h||_{H^{-1/2}(\Gamma)}. \tag{3.4}$$

Furthermore, by the application of stability theorems for the Petrov–Galerkin method, see [10, Theorem 5.3.1], and taking into account that the operators of multiplication with the functions $e^{\pm\langle b,x\rangle/\varepsilon}$ are continuously invertible on $H^{-1/2}(\Gamma)$, we obtain convergence of the Galerkin methods applied to the integral equation (2.6), too. Note, that the Galerkin method applied to the integral equation (2.5) is equivalent to the Petrov–Galerkin method applied to Eq. (2.6) with a modified right-hand side and an obvious modification of the approximating finite-dimensional spaces.

Remark 3.1. It is an open question, if estimate (3.4) is sharp.

4. Details about the implementation of the numerical schemes

In this paper we apply the Galerkin method with piece-wise constant trial functions on a regular mesh to the three integral equations (2.5)–(2.7) in the case $\Omega=(-1,1)\times(-1,1)$. Each of these integral equations can be rewritten by the use of a weakly singular integral operator A and an operator $K:H^{1/2}(\Gamma)\to H^{1/2}(\Gamma)$ in the form $Ag=Kf$. Note, that in Eqs. (2.5) and (2.6) we have that $K=I$, the identity operator, while in (2.7) we obtain that $K-1/2I$ is a strongly singular integral operator.

For given $N\in\mathbb{N}$ we divide each of the polygonal sides into N intervals of length $h=2/N$. Let χ_j, $j=1,\dots,4N$ be the characteristic functions of these intervals such that $\chi_1(-1+1/N,-1)=1$, and the characteristic functions being numbered in the mathematical positive sense. The Galerkin method is to solve the linear system

$$\sum_{l=1}^{4N}\alpha_l\int_\Gamma(A\chi_l)(x)\chi_j(x)\,\mathrm{d}s_x=\int_\Gamma(Kf)(x)\chi_j(x)\,\mathrm{d}s_x,\quad j=1,\dots,4N \tag{4.1}$$

for α_l. We shall outline some of the necessary details for the implementation of the integral equations. In the case that the supports of χ_l and χ_j are disjoint the double integral in the left-hand side of (4.1) was approximated by a direct application of Gauss-product rules of order 8×8. We also tried rules of order 16×16 without observing large differences in the range in which the numerical computations were done. The fundamental solution of the Helmholtz equation has been approximated by the use of a modification of a routine described in [8]. Care had to be taken during the computation of products of exponentially growing and exponentially decaying functions which appear in the fundamental solution of the convection–diffusion equation. To avoid overflow and other numerical instabilities, it is necessary to use a direct asymptotic expansion for the product

$$e^{\langle b,x-y\rangle/\varepsilon}K_0\left(\frac{|b|}{\varepsilon}|x-y|\right)$$

rather than to first compute the modified Bessel function by asymptotic expansion and post-multiplication with a possibly exponentially growing term. Furthermore, we took into account the following splitting of the fundamental solution for small numbers $t > 0$:

$$K_0(t) = \ln\frac{1}{t} I_0(t) + A_0(t), \tag{4.2}$$

I_0 and A_0 entire functions, see [1, p. 375, formulas 9.6.10–11] for the defintion of I_0 and A_0. The singular elements in the Galerkin matrices which can be represented as weakly singular integrals have been computed by the use of transformations to double integrals of the type

$$\int_{-1}^{1}\int_{0}^{1} \ln\frac{1}{t} f(t,s)\,\mathrm{d}t\,\mathrm{d}s,$$

f being smooth and the latter integrals are again approximated by Gauss-product rules of the order 16×8, with 16 Gauss-points for the logarithmic function obtained by Gautschi's program available from netlib and 8 Gauss–Legendre-points for the outer integration.

The strongly singular integral operator in (2.7) has fixed singularities in the corners of Ω. The numbers

$$r_{i,j} := \int_\Gamma \int_\Gamma \left\{ \partial_{n_y} K_0\left(\frac{|b|}{\varepsilon}|x-y|\right)\right\} \chi_i(y) \mathrm{e}^{-\langle b/\varepsilon, y\rangle}\,\mathrm{d}s_y \chi_j(x)\,\mathrm{d}s_x, \quad i,j = 1,\ldots,4N$$

vanish, if the support of χ_i and χ_j are on the same side of Ω. The computation of the numbers $r_{i,j}$ was done by the use of Gauss-quadrature if the integrand was nonsingular and for the corner elements we used integration by parts to reduce the computational problem to one for weakly singular integrals.

The approximation of the potential operators in (2.2)–(2.4) applied to the solutions of the discretised integral equations was done with high precision by the use of asymptotic expansions of the kernel functions and Gauss-product rules.

5. Numerical results

In this section we take $\Omega = (-1,1)\times(-1,1)$. We shall present numerical results for the test case $f = 1$ in Eq. (1.1) for $b = (1,0)$ and for $b = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ for varying ε. The true solution of (1.1) is $u = 1$ in that case. Nevertheless, we do study the behaviour of the solutions of the discretisations of the three different integral equations (2.5)–(2.7) by numerical computation of the potential operators in the grid points $x_{i,j} = (-1 + i/10, -1 + j/10)$, $i,j = 1,\ldots,19$. Then we compute the error $e_h = \max_{i,j=1,\ldots,19} |1 - u_h(x_{i,j})|$, with u_h being the Galerkin approximation of u.

In Tables 1–3 we give in column 1 the number ε, in column 2 the number of trial functions used on the boundary, in column 3 we give the error e_h, in column 4 the number of iterations in the iterative schemes which we used for the solution of the linear systems. We used the conjugate gradient method for the solution of the discretised integral equations (2.6) and (2.7) and we applied GMRES to the solution of (2.5), see [2,9] for the definition and the theory of the schemes. We did compare the results obtained with the results obtained by an application of a direct solver. For a large number of degree of freedom the iterative solvers have been faster and the results of the iterative solvers and that of the direct solver have been similar. The number of the table gives the equation number of the integral equation used.

Table 1
Integral equation (2.5)

ε	Dof	Error	GMRES-iterations
1d-1	160	3.43D-02	77
1d-1	240	1.85D-02	87
1d-1	320	7.14D-03	90
1d-1	400	4.87D-03	92

Table 2
Integral equation (2.6)

ε	Dof	Error	cg-iterations
1d-1	160	6.03D-03	82
1d-1	240	2.03D-03	102
1d-1	320	9.53D-04	116
1d-1	400	5.29D-04	130

Table 3
Integral equation (2.7)

ε	Dof	Error	cg-iterations
1d-1	160	4.21D-03	82
1d-1	240	3.68D-03	100
1d-1	320	2.45D-03	117
1d-1	400	1.58D-03	131

Table 4

ε	Error
1d-1	1.90D-03
8d-2	2.36D-03
6d-2	3.08D-03
4d-2	4.18D-03
2d-2	6.83D-03

In these simple test cases we obtained a good approximation of the true solution.

We obtained from Table 4 which again was computed for the case $f = 1$, $b = (1, 0)$ and we fixed the number dof $= 200$. We used the integral equation (2.5).

Obviously, the error e_h which is computed here, will be directly influenced by the Galerkin error and therefore we may conclude from the experiment that the Galerkin error grows when ε becomes smaller.

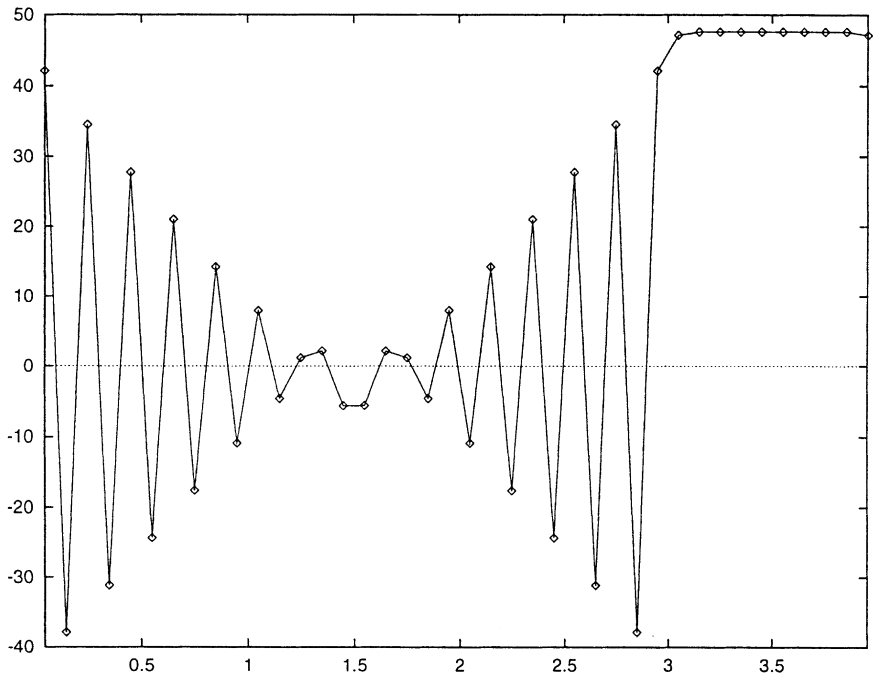

Fig. 1.

We present an example for the appearance of stability problems in case of smaller numbers ε. We take $\varepsilon = 0.02$, $b = (1,0)$ and we prescibe f by the boundary values of the function

$$u(x_1,x_2) = \frac{(e^{2x_1/\varepsilon} - e^{2/\varepsilon})}{(e^{-2/\varepsilon} - e^{2/\varepsilon})}, \tag{5.1}$$

which is then a solution of Eq. (1.1). Using the integral equation (2.7) we obtained an error

$$e_h = \max_{i,j=1,\ldots,19} \frac{|(u-u_h)(x_{i,j})|}{u(x_{i,j})} = 0.00227\ldots .$$

for the number of degree of freedom equal to 320. Again, the approximation seems to be satisfactory, at least for a sufficiently large number of degree of freedom. Taking the degree of freedom equal to 40 we visualise the error between the discretised solution g_h of the integral equation (2.7) versus the true solution g of this integral equation, i.e., g is the normal derivative of the function $v(x_1,x_2) = e^{-x_1/\varepsilon}u(x_1,x_2)$, with u defined in (5.1). Note, that we know the explicit solution of the integral equation (2.7) in this case, but we do not have explicit formulas for the solutions of (2.5) and (2.6). The squares in Fig. 1 mark the values of $(g_h - g)/|g_h - g|\log|g_h - g|$, at the mid-points of the 40 intervals which define the grid. To visualise the oscillatory behaviour of the discretised solution we used a log-scale and we connected the dotted values by lines.

We observe strong oscillations of the numerical solution for this low number of ansatz functions.

Finally, we show a more complicated example in which the data function f is not even in $H^{1/2}(\Gamma)$, but in $H^s(\Gamma)$ for all $s \in (0,1/2)$, namely, f is assumed to be piece-wise constant. We like

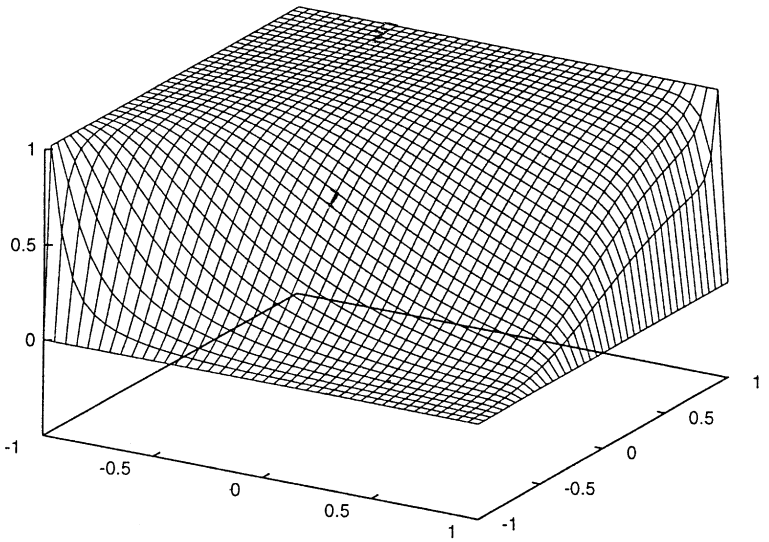

Fig. 2.

to remark here that the Galerkin and Galerkin–Petrov schemes using piece-wise polynomials are well defined for such functions f and the analysis of convergence can be done by the use of the inverse inequality and the regularity theorems for the Dirichlet problem on polygons. The invertibility of the single-layer integral operator as a mapping from $H^s(\Gamma)$ onto $H^{s+1}(\Gamma)$ for all $s \in (-1, -1/2)$ is well known, [3]. We conclude the paper with the plot of an interior layer computed with the aid of 640 piece-wise constant functions on Γ to solve Eq. (2.6) with $b = (\frac{\sqrt{3}}{2}, \frac{1}{2})$, $\varepsilon = 0.05$ and $f(-1, x_2) = 1 = f(x_1, 1)$, $f(x_1, -1) = 0 = f(1, x_2)$, if $|x_1|, |x_2| < 1$. The interior layer is clearly visible in Fig. 2.

Acknowledgements

This work was completed during the visit at the Instituto Superior Técnico. The author likes to thank the members of the Centro de Mathemática Aplicada for the invitation and for the opportunity to do common research.

References

[1] M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions, National Bureau of Standards, Washington, 1965.
[2] O. Axelsson, Itereative Solution Methods, Cambridge University Press, Cambridge, 1996.
[3] M. Costabel, Boundary integral operators on Lipschitz domains: elementary results, SIAM J. Math. Anal. 19 (1988) 613–626.
[4] G.I. Eskin, Boundary value problems for elliptic pseudodifferential operators, Transl. Math. Monographs, Vol. 52, American Mathematical Society, Providence, RI, 1981.
[5] P. Grisvard, Elliptic Problems in Nonsmooth Domains, Pitman, Boston, 1985.

[6] N.N. Lebedev, Special Functions and Their Applications, Dover, New York, 1972.
[7] J.L. Lions, Perturbations singulières dans les problemes aux limites et en controle optimal, Springer Lecture Notes in Mathematics, Vol. 323, Springer, Berlin, 1973.
[8] W.H. Press, S.A. Teukolsky, W.T. Vetterling, B.P. Flannery, Numerical Recipes, Cambridge University Press, Cambridge, 1992.
[9] Y. Saad, M.H. Schultz, GMRES: a generalized minimal residual algorithm for solving nonsymmetric linear systems, SIAM J. Sci. Statist. Comput. 7 (3) (1986) 856–869.
[10] A. Quarteroni, A. Valli, Numerical approximation of partial differential equations, Springer Series in Computational Mathematics, Vol. 23, 2nd Edition, Springer, Berlin, 1997.